Regularity property of noncommutative distributions: atoms and Atiyah property

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Embedding of free field and Atiyah property

Theorem (T. Mai, R. Speicher, Y., 18)

Let $X = (X_1, \ldots, X_n)$ be a tuple of selfadjoint random variables in a tracial W^*-probability space (M, τ) s.t. $\delta^\star(X_1, \cdots, X_n) = n$, then

1. *for any linear full $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$, $A(X)$ has no zero divisors;*
2. *for any $r \in \mathbb{C} \langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$, $r(X)$ is well-defined as an invertible unbounded operator;*
3. *for any $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$,*

$$\rho(A) = \operatorname{rank} A(X) := (\mathrm{Tr}_N \otimes \tau)(p_{\overline{\mathrm{im}\, A(X)}}).$$

Recall

A variant (non-microstates) free entropy dimension (A. Connes and D. Shlyakhtenko, 05):

$$\delta^\star(X_1, \cdots, X_n) := n - \liminf_{t \searrow 0} t\Phi^*(X_1 + \sqrt{t}S_1, \cdots, X_n + \sqrt{t}S_n).$$

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Remark

The second property is known for free groups. Namely, the free field can also be generated by the generators of free groups (P. Linnell, 93).

Embedding of free field and Atiyah property

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Let $X = (X_1, \ldots, X_n)$ *be a tuple of selfadjoint random variables in a tracial* W^**-probability space* (M, τ) *s.t.* $\delta^\star(X_1, \cdots, X_n) = n$*, then*

1. *for any linear full* $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$*,* $A(X)$ *has no zero divisors;*
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Recall

For a matrix $A \in M_N(\mathbb{C}\langle x_1, \cdots, x_n\rangle)$, its inner rank $\rho(A)$ is the least $r \in \mathbb{N}$ s.t. $\exists P \in M_{N,r}(\mathbb{C}\langle x_1, \cdots, x_n\rangle)$, $Q \in M_{r,N}(\mathbb{C}\langle x_1, \cdots, x_n\rangle)$ satisfying a factorization

$$A = PQ.$$

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Remark

If additionaly A is self-adjoint, then for any $\lambda \in \mathbb{R}$

$$1 - \frac{\operatorname{rank}(\lambda \mathbf{1}_N - A(X))}{N} = \mathrm{tr}_n \otimes \tau(p_{\ker(\lambda \mathbf{1}_N - A(X))}) = \mu_{A(X)}(\lambda)$$

Embedding of free field and Atiyah property

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Remark

These three properties are actually equivalent without assumption on X.

Strong Atiyah property

Definition

Let $X = (X_1, \ldots, X_n)$ be a tuple of operators in a finite von Neumann algebra (M, τ). If for any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n, x_1^*, \cdots, x_n^* \rangle)$,

$$\operatorname{rank} A(X) \in \mathbb{N} \cap [0, N],$$

then we say X has the *strong Atiyah property*.

Remarks

A tuple $X = (X_1, \cdots, X_n)$ has the strong Atiyah property if

- X is the tuple of generators of free groups (P. Linnell, 93).
- X is a tuple of freely independent normal random variables without atoms (D. Shlyakhtenko and P. Skoufranis, 15).

Strong Atiyah property

Definition

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Remarks

- Strong Atiyah property is equivalent to: for any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n, x_1^*, \cdots, x_n^* \rangle)$

  $$rank A(X) = \rho_{\mathcal{R}}(A(X))$$

  where $\mathcal{R}$ is the *rational closure* of X.

Application to block-structured random matrices

$X_{(N)}$ and $Y_{(N)}$ are two independent Wigner random matrices with Bernoulli distributed entries

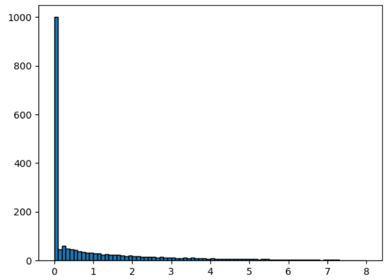
Let

$$A = \begin{pmatrix} y^2 & yxy \\ yxy & yx^2y \end{pmatrix} = \begin{pmatrix} y \\ yx \end{pmatrix} \begin{pmatrix} y & xy \end{pmatrix},$$

then

$\rho(A) = 1$ and $\rho(\lambda - A) = 2, \forall \lambda \neq 0.$

Histogram of eigenvalues of a sample $A(X_{(N)}, Y_{(N)})$ with $N = 1000$



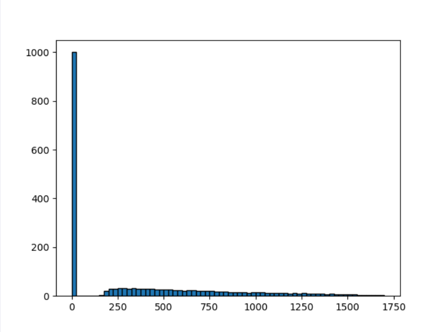
$X_{(N)}$ and $Y_{(N)}$ are two independent Wigner random matrices with Bernoulli distributed entries

Let

$$A = \begin{pmatrix} (y-5)^2 & (y-5)(x+5)(y-5) \\ (y-5)(x+5)(y-5) & (y-5)(x+5)^2(y-5) \end{pmatrix},$$

then

$\rho(A) = 1$ and $\rho(\lambda - A) = 2, \forall \lambda \neq 0.$



Central eigenvalues/atoms and where to find them

Definition (P. Cohn)

For a matrix $A \in M_N(\mathbb{C}\langle x_1, \cdots, x_n \rangle)$, $\lambda \in \mathbb{C}$ is called a *central eigenvalue* if $\lambda \mathbf{1}_N - A$ is not full, or equivalently, $\lambda \mathbf{1}_N - A$ is not invertible over $\mathbb{C}(\!\langle x_1, \cdots, x_n \rangle\!)$. Let $\sigma_c(A)$ denote the set of central eigenvalues of A.

Proposition (P. Cohn, 85)

For any $A \in M_N(\mathbb{C}\langle x_1, \cdots, x_n \rangle)$,

$$|\sigma_c(A)| \leq N.$$

Proposition (T. Mai, R. Speicher, Y., 18)

For any $A = A_0 + A_1 x_1 + \cdots + A_n x_n \in M_N(\mathbb{C}\langle x_1, \cdots, x_n \rangle)$,

$$\sigma_c(A) \subseteq \sigma(A_0).$$

Moreover, if $A - A_0$ is full, then $\sigma_c(A) = \emptyset$.

Thank you!